C 44605 (Pages : 2) Name.....................................

Reg. No.....................................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, AUGUST 2013

(CCSS)

Mathematics

MAT 2C 09—TOPOLOGY–I

Time : Three Hours

Maximum : 80 Marks

Part A

Answer all the questions
Each question carries 4 marks

1. Give an example of a Hausdorff topology that is not normal.
2. Prove that open balls in a metric space are open sets.
3. Define co-finite topology. Is it a Regular topology? Justify your answer.
4. Prove that the semi-open interval topology in the set of real numbers is stronger than its usual topology.
5. Is the real line with usual topology seperable? Justify your answer.
6. Find the derived set of the set Q of rational numbers in the real line with usual topology.
7. Prove that a metric space is a T_3 space.
8. Define divisible toplogical property. Give an example for the same.
9. Define extension problem and lifting problem in topological spaces.
10. Is the union of any two connected sets connected? Justify your claim.
11. Give an example of a wall in a product of sets.
12. Define large box in a topological space. Give an example of a large box.

Part B

Answer either A or B part of each question
Each question carries 8 Marks

13. A. (a) Determine the topology induced by discrete metric on a set.

(b) Define the Sierpinski space. Prove that this topology is not induced by a metric.

Turn over

B. (a) Consider two semi-open interval topologies on the set of real numbers. Prove that their meet is the usual topology while their join is the discrete topology on the set of real numbers.

(b) Prove that the topological product of a finite collection of second countable topological spaces is second countable.

14. A. (a) Prove that metrisability is a hereditary property.

(b) Prove that a subset A of a space X is dense in X if and only if for every non-empty open set B of X, $A \cap B \neq \phi$.

B. (a) For any set A in a space X, prove that the closure of A is the disjoint union of interior of A with the boundary of A.

(b) Prove that the closure of a connected set is connected.

15. A. (a) Prove that every closed surjective map is a quotient map.

(b) Prove that the composite of two quotient maps is a quotient map.

B. (a) Prove that the product of two connected spaces is connected.

(b) Prove that every continuous real valued function on a compact space is bounded and attains its bounds.

16. A. (a) Prove that a subset of the set of real numbers is connected if and only if it is an interval.

(b) In a Hausdorff space, prove that limits of sequences are unique.

B. (a) Prove that intersection of a finite. number of large boxes is a large box.

(b) Prove that the projection functions are open.

C 44602 (Pages : 2) Name..................................

Reg. No..................................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, AUGUST 2013

(CCSS)

Mathematics

MAT 2C 06—ALGEBRA – II

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 *marks.*

1. Construct a finite field of 8 elements
2. Let $a+bi \in C$ for $a+b \in R$ with $b \neq 0$. Show that $C = R(\alpha)$ where $\alpha = a+bi$.
3. Find the degree and a basis for $Q(\sqrt[3]{2}, \sqrt{3})$ over Q.
4. Find the number of primitive 18^{th} roots of unity in GF (19).
5. Define conjugate elements. Show that complex zeros of polynomials with real coefficients occur in conjugate pairs.
6. Find all conjugates of the number $\sqrt{1+\sqrt{2}}$ over Q.
7. Let E be the splitting field over Q of $x^3 - 3$ in Q [x]. Find the degree of E over Q.
8. Show that if $\alpha, \beta \in \overline{F}$ are both separable over F, then $\alpha \pm \beta, \alpha\beta$ and α/β, if $\beta \neq 0$, are all separable over F.
9. Describe the group of the polynomial $(x^4 - 1) \in Q[x]$ over Q.
10. Find over $\phi_8(x)$ over Z_3.
11. Show that the polynomial $x^5 - 1$ is solvable by radicals over Q.
12. Is it true that the Galois group of a finite extension of a finite field is solvable ? Justify your answer.

(12 × 4 = 48 marks)

Part B

Answer A **or** *B of each question.*
Each question carries 8 *marks.*

13. A (i) Let E be an extension field of F, and let $\alpha \in E$, where α is algebraic over F. Show that there is an irreducible polynomial $p(x) \in F[x]$ such that $p(\alpha) = 0$.

Turn over

(ii) Show that a field F is algebraically closed iff every non-constant polynomial in $F[x]$ factors in $F[x]$ into linear factors.

B (i) Show that if E is a finite extension field of a field F, and K is a finite extension field of E, then K is a finite extension of F, and $[K:F]=[K:E][E:F]$.

(ii) Prove that doubling the cube is impossible.

14. A (i) Prove that if F is a finite field of characteristic p, a prime, with algebraic closure $\overline{F}$, then $x^{p^n}-x$ has p^n distinct zeros in $\overline{F}$.

(ii) Describe all extensions of the identity map of Q to an iso-morphism mapping of $Q(\sqrt[3]{2})$ onto a subfield of $\overline{Q}$.

B (i) Let F be a finite field of characteristic a prime p. Show that the map $\sigma_p : F \to F$ defined by $\sigma_p(a)=a^p$ for $a \varepsilon F$ is an automorphism. Also, prove that $F_{\{\sigma_p\}} \cong Z_p$.

(ii) Let E be a field, and let F be a subfield of E. Show that the set G (E/F) of all automorphisms of E leaving F fixed forms a subgroup of the group of all automorphisms of E.

15. A (i) Define splitting field. Show that if $E \le \overline{F}$ is a splitting field over F, then every isomorphic mapping of E onto a subfield of $\overline{F}$ and leaving F fixed is an automorphism of E.

(ii) Show that if E is a finite extension of F, the E is separable over F iff each α in E is separable over F.

B (i) Show that every field of characteristic zero is perfect.

(ii) Prove that if E is an algebraic extension of a prefect field F, then E is perfect.

16. A Let K be the splitting field of (x^4+1) over Q. Describe the group G (K | Q). Give the lattice diagrams for the subfields of K and for the subgroups of G (K | Q).

B Let F be a field of characteristic zero, and let $a \varepsilon F$. Show that if K is the splitting field of $x^n - a$ over F, then G (K | F) is a solvable group.

(4 × 8 = 32 marks)

C 44604 (Pages : 3) Name..

Reg. No..

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, AUGUST 2013

(CCSS)

Mathematics

MAT 2C 08—ORDINARY DIFFERENTIAL EQUATIONS

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 *marks.*

1. Describe explicitly all inner products on R.

2. Apply the Gram-Schmidt process to the vectors $\beta_1 = (1, 0, 1)$, $\beta_2 = (1, 0, -1)$, $\beta_3 = (0, 3, 4)$, to obtain an orthonormal basis for R^3 with the standard inner product.

3. Show that any solution of the initial value problem $y' = f(x, y), y(x_0) = y_0$, where $f(x, y)$ is an arbitrary function defined and continuous in some neighbourhood of the point (x_0, y_0), is a continuous solution of the integral equation $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$ and vice-versa.

4. Find the general solution of $y'' - x f(x)y' + f(x)y = 0$.

5. Express $\sin^{-1}(x)$ in the form of a power series $\sum a_n x^n$ by solving $y' = (1-x^2)^{-½}$ in two ways.

6. Locate and classify the singular points on the x - axis for the differential equation

$$x^2(x^2-1)^2 y'' - x(1-x)y' + 2y = 0.$$

7. Find the general solution of the differential equation :

$(x^2-1)y'' + (5x+4)y' + 4y = 0$ near the singular point $x = -1$.

8. Obtain the recursion formula for the Legendre polynomials :

$$(n+1)\, P_{n+1}(x) = (2n+1)x\, P_n(x) - n\, P_{n-1}(x).$$

9. Show that $\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$.

10. Describe the phase portrait of the system : $\frac{dx}{dt} = +x, \frac{dy}{dt} = 0.$

Turn over

11. Show that a function of the form $ax^3 + bx^2y + cxy^2 + dy^3$ cannot be either positive definite or negative definite.

12. Show that (0, 0) is an asymptotically stable critical point of the system :

$$\frac{dx}{dt} = -y - x^3 \; ; \; \frac{dy}{dt} = x - y^3 .$$

(12 × 4 = 48 marks)

Part B

Answer (a) **or** *(b) of each question.*
Each question carries 8 marks.

13. (a) (i) Let W be a finite-dimensional subspace of an inner product space V and let E be the orthogonal projection of V on W. Show that E is an idempotent linear transformation of V onto W, $W^{\perp}$ is the null space of E, and $V = W \oplus W^{\perp}$.

(ii) Let W be the subspace of R^2 spanned by the vector (3, 4). Using the standard inner product, let E be the orthogonal projection of R^2 onto W. Find a formula for $E(x_1, x_2)$ and $W^{\perp}$.

(b) Solve the following initial value problem by Picard's method :

$$\frac{dy}{dx} = z, \quad y(0) = 1$$

$$\frac{dz}{dx} = -y, \quad z(0) = 0.$$

14. (a) Discuss the general solution of the homogeneous equation $y'' + py' + qy = 0$, where p and q are constants.

(b) Find two independent Frobenius series solutions of the equation $x^2y'' - x^2y' + (x^2 - 2)\,y = 0$.

15. (a) Derive Rodrigue's formula for Legendre polynomials and show that $P_n(x)$ given by the Rodrigue's formula satisfies the Legendre's equation $(1 - x^2)y'' - 2x\,y' + n(n+1)\,y = 0$, where n is a non-negative integer.

(b) Prove that $\int_0^1 x\, J_p(\lambda_m x)\, J_p(\lambda_n x) = \begin{cases} 0 & \text{if } m \neq n \\ \frac{1}{2} J_{p+1}(\lambda_n)^2 & \text{if } m = n, \end{cases}$ where λ_n's are the positive zeros of some fixed Bessel function $J_p(x)$ with $p \geq 0$.

16. (a) (i) Find the general solution of the system : $\frac{dx}{dt} = 5x + 4y,\ \frac{dy}{dt} = -x + y$.

(ii) Find the critical points and the differential equation of the path of the non-linear system :

$$\frac{dx}{dt} = y(x^2 + 1),\ \frac{dy}{dt} = 2xy^2.$$

(b) (i) Determine the nature and stability properties of the critical point (0, 0) for the system :

$$\frac{dx}{dt} = 5x + 2y,\ \frac{dy}{dt} = -17x - 5y.$$

(ii) Show that (0, 0) is an unstable critical point for the system :

$$\frac{dx}{dt} = 2xy + x^3,\ \frac{dy}{dt} = -x^2 + y^5.$$

($4 \times 8 = 32$ marks)

C 43789 (Pages : 3) Name..................................

Reg. No..................................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JULY 2013

(CCSS)

Mathematics

MAT 4E 06—FUNCTIONAL ANALYSIS—II

(2009 Admissions)

Time : Three Hours | Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

1. Let X denote the sequence space l^1. Let $\| \|'$ be a complete norm on X such that if $\|x_n - x\|' \to 0$ then $x_n^{(j)} \to x(j)$ for every $j = 1, 2, ...$ show that $\| \|'$ is equivalent to the usual norm $\| \|_1$ on X.

2. Let X be a normed space and $A \varepsilon BL(X)$. Show that A is invertible iff A is bounded below and surjective.

3. Show that every eigen space of a compact operator on a normed space X corresponding to a non-zero eigenvalue of A is infinite dimensional.

4. Let $\langle , \rangle$ be an inner product on a linear space X. Show that for all $x, y \varepsilon X, |\langle x, y\rangle|^2 \le \langle x, x\rangle \langle y, y\rangle$, where equality holds iff the set $\{x, y\}$ is linearly dependent.

5. Let $\{x_1, x_2,...\}$ be an orthogonal set in an inner product space X and $k_1, k_2,...$ be scalars having absolute value 1. Show that

$$\|k_1 x_1 + k_2 x_2 + ... + k_n x_n\| = \|x_1 + x_2 + ... + x_n\|.$$

6. Let X be an inner product space and $f \varepsilon X'$. Let $\{u_1, u_2,...\}$ be an orthonormal set in X. Show that $\sum_n |f(u_n)|^2 \le \|f\|^2$.

7. Let H be a Hilbert space and $A \varepsilon BL(H)$. Show that $\|A^*\| = \|A\|$ and $\|A^*A\| = \|A\|^2 = \|AA^*\|$.

8. Let $A \varepsilon BL(H)$ be self-adjoint. Show that $A^2 \ge 0$ and $A \le \|A\| \cdot I$.

(8 × 4 = 32 marks)

Turn over

Part B

Answer A **or** *B of each question.*
Each question carries 12 marks.

9. A. (i) Let X be a normed space and $A \varepsilon BL(X)$ be of finite rank. Show that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$.

(ii) Let X denote the sequence space l^2. Let $A : X \to X$ be defined by

$A(x) = \left(0, x(1), \frac{x(2)}{2}, \frac{x(3)}{3}, \ldots\right)$ for $x = (x(1), x(2), \ldots) \varepsilon X$. Determine $\sigma_e(A)$, $\sigma_a(A)$ and $\sigma(A)$.

B. (i) Let $1 \leq p < \alpha$ and $\frac{1}{p} + \frac{1}{q} = 1$. For a fixed $y \varepsilon l^q$, let $f_y(x) = \sum_{j=1}^{\alpha} x(j)\, y(j), x \varepsilon l^p$. Show that $f_y \varepsilon (l^p)'$, $\|f_y\| = \|y\|_q$ and the map $F : l^q \to (l^p)'$ defined by $F(y) = f_y ; y \varepsilon l^q$ is a linear isometry from l^q onto $(l^p)'$.

(ii) Show that if X is a finite dimensional normed space, then its dual X' has the same dimension as X.

10. A. (i) Let X and Y be normed spaces and $F : X \to Y$ be linear. Show that F is a compact map iff for every bounded sequence (x_n) in X, $(F(x_n))$ has a subsequence which converges in Y.

(ii) Let X and Y be normed spaces and $F \varepsilon BL(X, Y)$. Define the Transpose of F and show that if F is compact then the Transpose of F is also compact.

B. (i) Let X be a linear space, $A : X \to X$ linear and $A(x_n) = k_n\, x_n$ for some $0 \neq x_n \varepsilon X$ and $k_n \varepsilon k$ with $k_n \neq k_m$ whenever $n \neq m ; n = 1, 2, \ldots$. Show that $\{x_1, x_2, \ldots\}$ is linearly independent subset of X.

(ii) Show that every inner product space is a normed space.

11. A. (i) State and prove Bessel's inequality.

(ii) Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Show that $\{u_\alpha\}$ is an orthonormal basis for H iff span $\{u_\alpha\}$ is dense in H.

B. (i) Let H be a Hilbert space, G be a subspace of H and $g \varepsilon G'$. Show that there is a unique $f \varepsilon H'$ such that $f / G = g$ and $\|f\| = \|g\|$.

(ii) Let H be a Hilbert space and $A \in BL(H)$. Show that there is a unique $B \in BL(H)$ such that for all $x, y \in H$, $\langle A(x), y \rangle = \langle x, B(y) \rangle$.

12. A. (i) Let H be a Hilbert space and $A \in BL(H)$. Show that A is unitary iff $\|A(x)\| = \|x\|$ for all $x \in H$ and A is surjective.

(ii) Let A be a self-adjoint operator on a finite dimensional Hilbert space H. Show that every root of the characteristic polynomial of A is real.

B. (i) Let $A \in BL(H)$ be compact. Show that A^* is compact.

(ii) Let $A \in BL(H)$. Show that A is compact iff A^*A is compact.

($4 \times 12 = 48$ marks)

C 43790 (Pages : 2) Name..................................

Reg. No..................................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JULY 2013

(CCSS)

Mathematics

MAT 4E 07—ALGEBRAIC TOPOLOGY

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 *marks.*

1. Define a geometric complex and give an example.
2. Define the p^{th} incidence matrix of an oriented complex K. Compute the 2^{nd} incidence matrix of the closure of a 3-simplex $\sigma^3 = < a_0\ a_1\ a_2\ a_3 >$ with vertices ordered by $a_0 < a_1 < a_2 < a_3$.
3. Prove that C_p (K), the family of p-chains on an oriented complex K, forms a group with the operation of pointwise addition induced by integers.
4. If S is a simple polyhedron with V vertices, E edges and F faces, then prove that $V - E + F = 2$.
5. Define a simplicial mapping. Give an example of a simplicial mapping.
6. State and prove the Brouwer fixed point theorem.
7. Prove that every contractible space is simply connected.
8. Prove that the fundamental group of the punctured plane is isomorphic to the group Z of integers under addition.

(8 × 4 = 32 marks)

Part B

Answer **either** *A* **or** *B of each question.*
Each question carries 16 *marks.*

9. A (a) Define a geometrically independent set in R^n. Prove that a set $\{a_0, a_1, \ldots, a_k\}$ of points in R^n is geometrically independent iff the set of vectors $\{a_1 - a_0, a_2 - a_0, \ldots, a_k - a_0\}$ is linearly independent.

   (b) Let K be an oriented complex, σ^p an oriented p-simplex of K and σ^{p-2} a $(p-2)$- face of σ^p . Prove that $\sum\left[\sigma^p, \sigma^{p-1}\right]\left[\sigma^{p-1}, \sigma^{p-2}\right] = 0, \sigma^{p-1} \in K$.

Turn over

B (a) Prove that, for the projective p, $H_1(p) \cong Z_2$, the group of integers modulo 2.

(b) Let K be a complex with r combinatorial components. Prove that $H_0(K)$ is isomorphic to the direct sum of r copies of the group Z of integers.

10. A (a) Define a regular polyhedron. Prove that there are only five regular simple polyhedra.

(b) Prove that an n-pseudomanifold K is orientable if and only if the n^{th} homology group $H_n(K)$ is not the trivial group.

B (a) Define a chain mapping between two complexes and show that it induces homomorphism between the homology groups in each dimension.

(b) Prove that , if two continuous maps $f, g : s^n \to s^n$ are homotopic, then they have the same degree.

11. A (a) Describe how one can associate a group with a topological space using the idea of loops and homotopy.

(b) Prove that the set $\pi_1(X, x_0)$ is a group under the 0 operation.

B (a) If A is a deformation retract of a space X and x_0 is a point of A, prove that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(A, x_0)$.

(b) Prove that two loops α and β in S^1 with basepoint 1 are equivalent if and only if they have the same degree.

($3 \times 16 = 48$ marks)

C 43791 (Pages : 2) Name.......................................

Reg. No....................................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JULY 2013

(CCSS)

Mathematics

MAT 4E 08—GRAPH THEORY

(2009 Admissions)

Time : One Hour and a Half Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 8 *marks.*

1. Let G be a graph, L(G) be its line graph and let $k(\mathrm{G}) = k$. Is $k(\mathrm{L}(\mathrm{G})) = k$? Justify your answer.

2. Let $b(v)$ denote the number of blocks of a simple connected graph G to which a vertex v belongs. Prove that the number of blocks $b(\mathrm{G})$ of G is given by :

$$b(\mathrm{G}) = 1 + \sum_{v \in \mathrm{V(G)}} (b(v) - 1).$$

3. Prove that in a critical graph G, no vertex cut is a clique.

4. Prove that the chromatic polynomial of a wheel with n vertices is $\lambda(\lambda - 2)^n + (-1)^n \lambda(\lambda - 2)$.

(4 × 8 = 32 marks)

Part B

Answer A **or** *B of each question.*
Each question carries 24 *marks.*

5. A. (i) Prove that a graph G with atleast three vertices is 2-connected if and only if any two vertices of G are connected by atleast two internally disjoint paths.

(ii) Prove that a connected simple graph G is 3-edge connected if and only if every edge of G is the intersection of the edge sets of two cycles of G.

B. (i) Prove that in any network N, the value of any flow f is less than or equal to the capacity of any cut K.

(ii) Determine the values of the parameters α, α', β and β' for the Peterson graph.

Turn over

6. A. (i) Define critical graphs. Prove that a critical graph is connected.

(ii) If a connected graph G is neither an odd cycle nor a complete graph, then prove that $\chi(G) \leq \Delta(G)$.

B. (i) Let G be a loopless bipartite graph. Prove that $\chi'(G) = \chi(G)$.

(ii) Let G be a simple graph of order n and size m. Prove that :

1 $f(G;\lambda)$ is a monic polynomial of degree n in λ with integer coefficients and constant term zero.

2 Coefficient of $f(G;\lambda)$ are alternate in sign and the coefficient of λ^{n-1} is $-m$.

(2 × 24 = 48 marks)

C 44603

(Pages : 3)

Name.....................................

Reg. No.................................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, AUGUST 2013

(CCSS)

Mathematics

MAT 2C 07—REAL ANALYSIS—II

Time : Three Hours

Maximum : 80 Marks

Part A

Answer all questions. Each question carries 4 marks

I. (1) Let X be a vector space and let $\dim X = n$. Prove that a set E of n vectors in X spans X if and only if E is independent.

(2) Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$. Prove that $\|A\| < \infty$ and A is a uniformly continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^m$.

(3) If f and g are differentiable real functions in $\mathbb{R}^n$, then prove that

$$\Delta(fg) = f\Delta g + g\Delta f.$$

(4) If A is a countable subset of $\mathbb{R}$, then prove that $m^*(A) = 0$.

(5) If E_1 and E_2 are measurable sets, then prove that $E_1 \cup E_2$ is a measurable set.

(6) Show that if f is a measurable real-valued function and g a continuous function defined on $(-\infty, \infty)$, then $g \circ f$ is measurable.

(7) Let E be a measurable set of finite measure and let $f = \chi_E$, the characteristic function of E. Prove that f is measurable and evaluate $\int f$.

(8) Let f and g be bounded measurable functions defined on a set E of finite measure and let $f = g$ *a.e.*. Prove that $\int f = \int g$.

(9) Define convergence in measure. Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E of finite measure such that $f_n \to f$ *a.e.*. Prove that $\{f_n\}$ converges to f in measure.

(10) If $f'(x)$ exists, then prove that $D^+(f+g)(x) = f'(x) + D^+g(x)$.

(11) Show that if f is a real valued function defined on $[a, b]$, f' exists and bounded on $[a, b]$, then f is of bounded variation on $[a, b]$.

(12) Define absolute continuity and give an example of it. Prove that sum of two absolutely continuous functions is absolutely continuous.

Part B

Answer A or B of each question.
Each question carries 8 marks

II. A (a) Prove that a linear operator A on a finite dimensional vector space X is one-to-one if and only if A is onto.

Turn over

(b) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$, then prove that $\|AB\| \leq \|A\|\|B\|$.

(c) Let f map a convex open subset E of $\mathbb{R}^n$ into $\mathbb{R}^m$ and let f be differentiable in E. If $f'(x) = 0$ for all $x \in E$, then prove that f is a constant.

B (a) Let f map an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Prove that f is continuously differentiable in E if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m$, $1 \leq j \leq n$.

(b) Define contraction mapping. If X is a complete metric space and if φ is a contraction of X into X, then prove that there exists one and only one $x \in X$ such that $\varphi(x) = x$.

III. A (a) Let $\{E_n\}$ be an infinite decreasing sequence of measurable sets and let $m(E_1)$ is finite. Prove that

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} m(E_n).$$

(b) Prove that the set of all Lebesgue measurable sets is a σ-algebra.

B (a) Let $\{E_i\}$ be a sequence of disjoint measurable sets and A be any set. Prove that

$$m^*\left(A \cap (\bigcup_{i=1}^{\infty} E_i)\right) = \sum_{i=1}^{\infty} m(A \cap E_i).$$

(b) Prove that Cantor set is of measure zero.

(c) Let $\{f_n\}$ be a sequence of measurable functions. Prove that $\inf_n f_n$ is a measurable function.

IV. A (a) Let f be a bounded measurable function and let A and B are disjoint measurable sets of finite measure. Prove that

$$\int_{A \cup B} f = \int_A f + \int_B g.$$

(b) Give an example of a sequence of measurable functions where strict inequality occurs in Fatou's lemma.

(c) Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions and let $f = \lim f_n$ *a.e.*. Prove that

$$\int f = \lim \int f_n.$$

B (a) State and prove Lebesgue convergence theorem.

(b) Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E of finite measure. If $\{f_n\}$ converges to f in measure, then prove that there is a subsequence $\{f_{n_k}\}$ that converges almost everywhere to f.

V. A (a) If f is continuous on $[a, b]$ and one of its derivatives is everywhere non-negative on (a, b), then prove that f is non-decreasing on $[a, b]$.

(b) If f is integrable and $\int_a^x f(t)\,dt = 0$ for all $x \in [a, b]$, then prove that $f(t) = 0$ a.e. in $[a, b]$.

(c) If f is absolutely continuous, then prove that f has derivative almost everywhere.

B (a) If f is of bounded variation on $[a, b]$, then prove that $f'(x)$ exists for almost all x in $[a, b]$.

(b) Prove that a function F is an indefinite integral if and only if it is absolutely continuous.