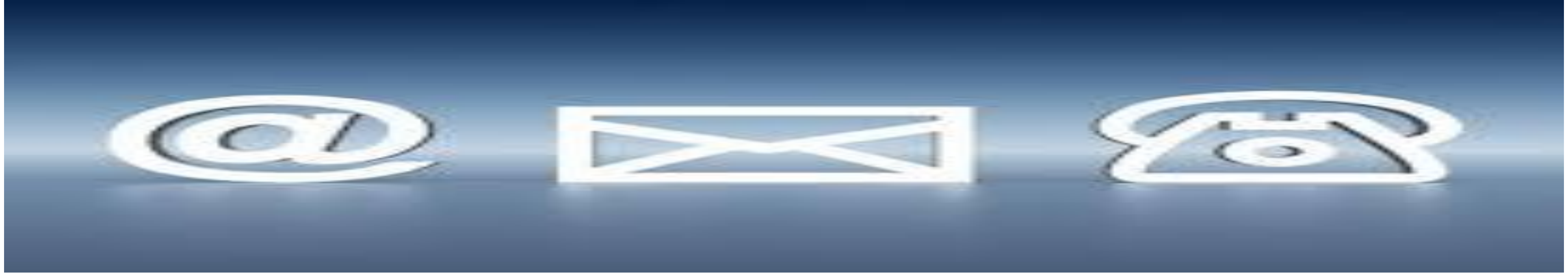

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PHILOSOPHY OF MATHEMATICS

Lecture 3

LAW OF EXCLUDED MIDDLE

Every statement is either true or false but not both.

MANY VALUED LOGICS

In logic, a **many-valued logic** (also **multi-** or **multiple-valued logic**) is a propositional calculus in which there are more than two truth values. Traditionally, in Aristotle's logical calculus, there were only two possible values (i.e., "true" and "false") for any proposition. Classical two-valued logic may be extended to ***n*-valued logic** for *n* greater than 2. Those most popular in the literature are three-valued (e.g., Łukasiewicz's and Kleene's, which accept the values "true", "false", and "unknown"), the finite-valued (finitely-many valued) with more than three values, and the infinite-valued (infinitely-many valued), such as fuzzy logic and probability logic.

IGNORANCE AND KNOWLEDGE

Without ignorance, there would be no knowledge, without knowledge, there would be no ignorance.

SET THEORIES

- Fuzzy Set Theory
- Rough Set Theory
- Soft Set Theory
- Multi Sets
- Nonstandard Set Theory
- Alternative Set Theory

PEANO'S AXIOMS

1. 0 is a number
2. The successor of any number is a number
3. No two numbers have the same successor
4. 0 is not the successor of any number
5. Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers

(Induction Axiom)

INDUCTION FAILS

An assertion may be valid in an entire series of particular cases, and at the same time invalid in general

PRINCIPLE OF MATHEMATICAL INDUCTION

P(n) for n= 1,2,3…

P(1) is true

P(k) implies P(k+1) for all k

Then P(n) is true for all n=1,2,3…

AXIOMATIC METHOD

EULID

- Elements (c. 300 B.C.E.)
- Axioms- Common Notions- self evident truth
- Postulates-a geometrical fact so simple and obvious that its validity may be assumed
- From these Euclid deduced 465 Propositions in a logical chain.

ARISTOTLE (381-321 B.C.E.)

- Every demonstrative science must start from in demonstrable principles; otherwise, the steps of demonstration would be endless. Of these indemonstrable principles some are (a) common to all sciences, others are (b) particular, or peculiar to the particular science; (a) the common principles are the axioms, most commonly illustrated by the axiom that, if equals be subtracted from equals, the remainders are equal. In (b) we have first the genus or the subject – matter, the existence of which must be assumed

UNDEFIND TERMS

AXIOMS

DEFINITIONS

DEFINITION OF DEFINITION

A definition is an agreement to substitute a single term or symbol for more complex terms or symbols

Veblan O.

A System of Axioms for Geometry, Trans. Amer. Math. Soc. Vol.5(1904) 343-384

Undefined terms defined indirectly through axioms

EXAMPLE

Undefined Terms

Point, Line

EXAMPLE: AXIOMS

1. Every line is a collection of points.
2. There exists at least two points.
3. If p and q are points, there exists one and only one line containing p and q
4. If L is a line, there exists a point not on L.
5. If L is a line, and p is a point not on L, then there exists one and only one line containing p that is parallel to L

FIFTH POSTULATE OF EUCLID

If a straight line falling on two straight likes make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles less than two right angles

PLAYFAIR AXIOM

Given a line L and a point P not on L, there exists one and only one line containing P and parallel to L

John Playfair (1748-1819)

FIFTH POSTULATE OF EUCLID

Fifth Postulate can neither be proved nor disproved from other axioms

NONEUCLIDEAN GEOMETRIES

Janos Bolyai (1802-1860)

Nikolai Lobachevski (1792-1856)

Carl Friedrich Gauss (1777-1855)

Bernhard Riemann (1826-1866)

Hyperbolic Geometry of Lobachevski (al least two lines)

Elliptic Geometry of Riemann(No line)

CONSISTENCY

A contradiction cannot be deduced from the axioms

INDEPENDENCE

None of the axioms cannot be proved using the other axioms

COMPLETENESS

Any statement is either proved or disproved using the axioms

KURT GODEL

1906-1978

A nontrivial axiom system for mathematics cannot prove its consistency

In any nontrivial axiom system for mathematics contains a statement which can not be proved and which can not be disproved

RUSSELL'S PARADOX

Let S be the set of all sets. A set A is said to be a good set if A is not an element of A. Let

G= The set of all good sets = { A $\in$ S: A is not an element of A}

Then G is an element of S.

If G is an element of G , then G is not an element of G.

If G is not an element of G, then G is an element of G.

Contradiction in any case.

BARBER PARADOX

The barber is the "one who shaves all those, and those only, who do not shave themselves". The question is, does the barber shave himself?

Answering this question results in a contradiction. The barber cannot shave himself as he only shaves those who do not shave themselves. Thus, if he shaves himself he ceases to be the barber. Conversely, if the barber does not shave himself, then he fits into the group of people who would be shaved by the barber, and thus, as the barber, he must shave himself.

It was used by Bertrand Russell himself as an illustration of the paradox, though he attributes it to an unnamed person who suggested it to him

BURALI-FORTI PARADOX

In set theory, a field of mathematics, the **Burali-Forti paradox** demonstrates that constructing "the set of all ordinal numbers" leads to a contradiction and therefore shows an antinomy in a system that allows its construction. It is named after Cesare Burali-Forti, who in 1897 published a paper proving a theorem which, unknown to him, contradicted a previously proved result by Cantor. Bertrand Russell subsequently noticed the contradiction, and when he published it in his 1903 book *Principles of Mathematics*, he stated that it had been suggested to him by Burali-Forti's paper, with the result that it came to be known by Burali-Forti's name.

CESARE BURALI-FORTI

Cesare Burali-Forti (13 August 1861 – 21 January 1931) was an Italian mathematician, after whom the Burali-Forti paradox is named.

Burali-Forti was born in Arezzo, and was an assistant of Giuseppe Peano in Turin from 1894 to 1896, during which time he discovered a theorem which Bertrand Russell later realised contradicted a previously proved result by Georg Cantor. The contradiction came to be called the Burali-Forti paradox of Cantorian set theory. He died in Turin.

AXIOMATIC SET THEORY

Zermelo–Fraenkel set theory (ZF)

Gödel-Bernays Set Theory

Morse- Kelly Set Theory

AXIOM OF CHOICE

Given any nonempty collection of disjoint nonempty sets, there exists a set containing exactly one element of each set of the collection.

E. Zermelo 1904 (to prove that any set can be well ordered)

The arbitrary product of a nonempty collection of nonempty sets is nonempty.

SOME CONSEQUENCES

- Any ideal of a commutative ring with unity is contained in a maximal ideal.
- Any vector space has a basis.
- Arbitrary product of compact topological spaces is compact.(This is also equivalent to the Axiom of Choice)
- Any infinite set contains a countably infinite set.
- If x is a limit point of a subset A of the real line, there exists a sequence of elements of A converging to x.
- The Hahn- Banach Theorem.
- The Stone-Cech Compactification of a Tychonoff space.
- There exists a nonmeasurable set.

EQUIVALENTS OF THE AXIOM OF CHOICE

- Zorn’s Lemma
- Well ordering Theorem
- Hausdorff Maximality Principle
- Tukey’s Lemma

EQUIVALENTS OF THE AXIOM OF CHOICE

H. Rubin & J.E. Rubin

BANACH TARSKI PARADOX (1924)

The **Banach–Tarski paradox** is a theorem in set-theoretic geometry, which states the following: Given a solid ball in 3-dimensional space, there exists a decomposition of the ball into a finite number of disjoint subsets, which can then be put back together in a different way to yield two identical copies of the original ball. Indeed, the reassembly process involves only moving the pieces around and rotating them without changing their shape. However, the pieces themselves are not "solids" in the usual sense, but infinite scatterings of points. The reconstruction can work with as few as five pieces.

The reason the Banach–Tarski theorem is called a paradox is that it contradicts basic geometric intuition.

- K. Godel 1938 : The Axiom of Choice cannot be disproved.

  If ZF is consistent, then ZFC is also consistent

- P.J. Cohen 1963 : The Axiom of Choice cannot be proved

  AC is independent of ZF

The Axiom of Choice is neither true nor false.

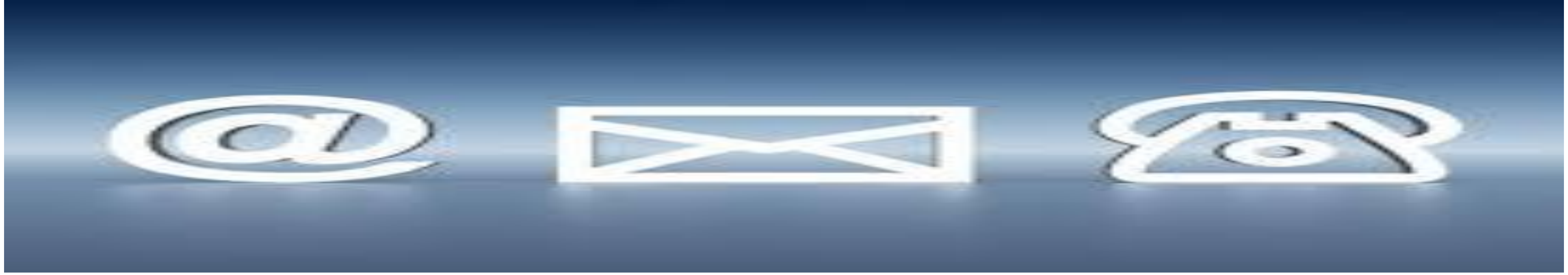
One class is similar to another class if there is a bijection between them i. e. a one to one correspondence

- The number of a class is the class of all those classes that are similar to it
- A number is anything which is the number of some class

RAYMOND L. WILDER

Introduction to the Foundations of Mathematics

John Wiley & Sons, Inc. New York 1952



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